

ON THE EXISTENCE OF A PROJECTIVE RECONSTRUCTION

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ABSTRACT. In this note we study the connection between the existence of a projective reconstruction and the existence of a fundamental matrix satisfying the epipolar constraints.

1. INTRODUCTION

Let a set of point correspondences $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ ($i = 1, \dots, m$) be given. Consider the following three statements:

- (A) (x_i, y_i) are the images of m points in \mathbb{R}^3 in two uncalibrated cameras.
- (B) (x_i, y_i) are the images of m points in \mathbb{P}^3 in two uncalibrated cameras.
- (C) There exists a fundamental matrix F such that the (x_i, y_i) satisfy the epipolar constraints.

A in-depth study of (C) can be found in [1]. The goal of this note is to understand the connection among these three statements. In the following we summarize our contribution. All the results are proved using just linear algebra.

- (1) A standard result in two-view geometry [2, §9.2] states that (A) implies (C). In [2] this result was proved by classical projective geometry and drawing pictures. We offer a modern, more rigorous, and linear algebraic proof; see Theorem 4.1.
- (2) It is clear that (A) implies (B). We will show (A) and (B) are indeed equivalent; see Theorem 3.1. The proof is based on constructing an appropriate projective transformation.
- (3) We show that (C) implies (A) after making an additional assumption about the point pairs (x_i, y_i) . Indeed, if (C) holds, one can construct a pair of uncalibrated cameras P_1, P_2 associated to F . If we assume that x_i is an epipole of P_1 if and only if y_i is an epipole of P_2 , then (A) holds. This assumption is also necessary for (A) to hold. As a result, we know (A) holds if and only if (C) and this assumption hold. This is the main theorem of this note; see Theorem 4.6.

In Section 2 we introduce the notation and definitions that will be used. In Section 3 we discuss projective reconstruction using finite, infinite, coincident and non-coincident cameras. Finally we provide a proof of the main theorem using linear algebra, in Section 4.

2. NOTATION AND DEFINITIONS

To begin with, we introduce the notation and definitions that will be used in this note; see [2].

Denote the n -dimensional real projective space by \mathbb{P}^n . For any $x, y \in \mathbb{P}^n$, we say $x \sim y$ if there exists $\lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$. The set of $m \times n$ matrices with

entries in \mathbb{R} is denoted by $\mathbb{R}^{m \times n}$, and by $\mathbb{P}^{m \times n}$ if the matrices are only up to scale. For $v \in \mathbb{R}^3$,

$$[v]_{\times} := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is a skew-symmetric matrix whose rank is two unless $v = 0$. Also, $[v]_{\times} w = v \times w$, where \times denotes the vector cross product. For any $x \in \mathbb{R}^n$ the symbol \hat{x} denotes $(x, 1)^{\top}$ in \mathbb{P}^n . A point in \mathbb{P}^n is called *finite* if it can be identified with $(x, 1)^{\top}$ for some $x \in \mathbb{R}^n$.

A (*projective*) *camera* can be modeled by a matrix $P \in \mathbb{P}^{3 \times 4}$ with $\text{rank}(P) = 3$. Partitioning a camera as $P = \begin{pmatrix} A & b \end{pmatrix}$ where $A \in \mathbb{R}^{3 \times 3}$, we say that P is a *finite camera* if A is nonsingular. The camera center of P is $(-A^{-1}b, 1)^{\top}$ if P is finite; and $(w, 0)^{\top}$ otherwise, where w lies in the kernel of A . Two cameras P_1, P_2 with camera centers c_1, c_2 are *coincident* if $c_1 \sim c_2$. A tuple $(P_1, P_2, \{w_i\}_{i=1}^m)$ is called a (*projective*) *reconstruction* of $\{(x_i, y_i)\}_{i=1}^m \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ if P_1 and P_2 are projective cameras, $w_i \in \mathbb{P}^3$ and

$$P_1 w_i \sim \hat{x}_i, \quad P_2 w_i \sim \hat{y}_i \quad \text{for all } i = 1, \dots, m.$$

If in addition, P_1, P_2 are finite cameras and w_i are finite points for all i , then $(P_1, P_2, \{w_i\})$ is called a *finite (projective) reconstruction*.

A real 3×3 matrix F is a *fundamental matrix* associated to $\{(x_i, y_i)\}$ if F has rank two and the following *epipolar constraints* hold:

$$\hat{y}_i^{\top} F \hat{x}_i = 0 \quad \text{for any } i.$$

3. PROJECTIVE RECONSTRUCTION

Given point correspondences $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2, i = 1, \dots, m\}$, the *projective reconstruction problem* is to decide if there is a projective reconstruction of these point pairs, and the *finite projective reconstruction problem* is to determine if the pairs admit a finite projective reconstruction. We first show that these two problems, as well as two others that naturally interpolate between them, are all equivalent.

Theorem 3.1. *Let $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2, i = 1, \dots, m\}$ be given. Then the following statements are equivalent:*

- (1) *There are cameras P_1, P_2 and points $w_i \in \mathbb{P}^3, i = 1, \dots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*
- (2) *There are FINITE cameras P_1, P_2 and points $w_i \in \mathbb{P}^3, i = 1, \dots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*
- (3) *There are FINITE cameras P_1, P_2 and FINITE points $w_i \in \mathbb{P}^3, i = 1, \dots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$.*
- (4) *There is a FINITE camera P_2 and FINITE points $w_i \in \mathbb{P}^3, i = 1, \dots, m$, such that $(P_1, P_2, \{w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$, with the first camera $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ where I is the 3×3 identity matrix.*

If P is a camera matrix, there is a nonsingular matrix $H \in \mathbb{R}^{4 \times 4}$ such that $PH^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$. For instance, take H to be the nonsingular 4×4 matrix obtained by adding an appropriately chosen additional row to P . In order to prove Theorem 3.1, we will first need the following simple fact that for any finite collection of nonzero points in \mathbb{R}^n , there is always a hyperplane through the origin that avoids all of them.

Lemma 3.2. *Given $v_1, \dots, v_m \in \mathbb{R}^n \setminus \{0\}$, there exists $a \in \mathbb{R}^n$ such that $a^\top v_i \neq 0$ for all i .*

Proof: Let $S := \{v_1, \dots, v_m\}$. We want to show that there exists $a \in \mathbb{R}^n$ such that $a^\perp \cap S = \emptyset$. Suppose to the contrary, for any $a \in \mathbb{R}^n$ one has $a^\perp \cap S \neq \emptyset$. Then $a \in v_i^\perp$ for some i . Thus $\mathbb{R}^n = v_1^\perp \cup \dots \cup v_m^\perp$ which implies that $\mathbb{R}^n = v_i^\perp$ for some i , and hence, this $v_i = 0$. This contradicts our assumption. \square

We now come to the key ingredient in the proof of Theorem 3.1 which allows us to always replace a projective reconstruction with a finite projective reconstruction whenever the first camera is of the form $(I \ 0)$.

Lemma 3.3. *Given point pairs $\{(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2, i = 1, \dots, m\}$, suppose we have cameras $P_1 = (I \ 0)$ and $P_2 = (A \ b)$, a set $\sigma \subseteq \{1, \dots, m\}$, and points $v_i \in \mathbb{R}^3, i = 1, \dots, m$ such that:*

$$\forall i \in \sigma, \quad v_i \neq 0, \quad P_1 \begin{pmatrix} v_i \\ 0 \end{pmatrix} \sim \hat{x}_i \quad \text{and} \quad P_2 \begin{pmatrix} v_i \\ 0 \end{pmatrix} \sim \hat{y}_i;$$

$$\forall i \notin \sigma, \quad P_1 \hat{v}_i \sim \hat{x}_i \quad \text{and} \quad P_2 \hat{v}_i \sim \hat{y}_i.$$

Then there exists a finite camera P'_2 and points $v'_i \in \mathbb{R}^3, i = 1, \dots, m$ such that $(P_1, P'_2, \{\hat{v}'_i\})$ is a finite reconstruction of $\{(x_i, y_i)\}$. In addition, if $b \neq 0$, then P_1 and P'_2 are non-coincident cameras.

Proof: Let the camera centers of P_1 and P_2 be represented by $c_1 = \hat{0}$ and c_2 respectively. Since $c_1, c_2, (v_i^\top, 0)^\top, i \in \sigma$ and $\hat{v}_i, i \notin \sigma$ are all nonzero points in \mathbb{R}^4 , by Lemma 3.2 there is a vector $a \in \mathbb{R}^3$ and a scalar $\alpha \in \mathbb{R}$ such that

$$(3.1) \quad (a^\top \alpha) c_i \neq 0, i = 1, 2, \quad (a^\top \alpha) \begin{pmatrix} v_i \\ 0 \end{pmatrix} \neq 0 (i \in \sigma), \quad (a^\top \alpha) \hat{v}_i \neq 0 (i \notin \sigma).$$

Since $(a^\top \alpha) c_1 \neq 0$, we have that $\alpha \neq 0$. So by scaling, we may assume that $\alpha = 1$ in (3.1).

Consider the invertible matrix $H := \begin{pmatrix} I & 0 \\ a^\top & 1 \end{pmatrix}$. Then $H^{-1} := \begin{pmatrix} I & 0 \\ -a^\top & 1 \end{pmatrix}$, and $P_1 H^{-1} = P_1$ and $P_2 H^{-1} = (A - ba^\top \ b)$. Furthermore,

$$H c_2 = \begin{pmatrix} * \\ (a^\top \ 1) c_2 \end{pmatrix}, \quad H \begin{pmatrix} v_i \\ 0 \end{pmatrix} = \begin{pmatrix} v_i \\ (a^\top \ 1) \begin{pmatrix} v_i \\ 0 \end{pmatrix} \end{pmatrix}, \quad H \hat{v}_i = \begin{pmatrix} v_i \\ (a^\top \ 1) \hat{v}_i \end{pmatrix}$$

which are all finite by (3.1). In particular, $P_2 H^{-1}$ is a finite camera as its center $H c_2$ is finite. The proof is completed by taking $P'_2 = P_2 H^{-1}$, $\hat{v}'_i \sim H \begin{pmatrix} v_i \\ 0 \end{pmatrix} (i \in \sigma)$ and $\hat{v}'_i \sim H \hat{v}_i (i \notin \sigma)$.

If we further assume $b \neq 0$, then P_1 and P_2 are non-coincident cameras. Hence $P_1 = P_1 H^{-1}$ and $P'_2 = P_2 H^{-1}$ are also non-coincident. \square

Proof of Theorem 3.1: Clearly, (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1). For (1) \Rightarrow (4), let H be a homography so that $P'_1 := P_1 H^{-1} = (I \ 0)$ and let $P'_2 := P_2 H^{-1} = (A \ b)$. Then $(P'_1, P'_2, \{H w_i\})$ is a reconstruction of $\{(x_i, y_i)\}$. We can now use Lemma 3.3 to turn this into a finite reconstruction where the first camera is still $(I \ 0)$. Therefore, we conclude that all four statements in the theorem are equivalent. \square

We now prove that the equivalences in Theorem 3.1 also hold if we further require that the cameras are non-coincident (coincident) in each statement.

Theorem 3.4. *The four statements in Theorem 3.1 are equivalent if we replace “cameras P_1, P_2 ” in each statement with “non-coincident cameras P_1, P_2 ”.*

Proof: As before, we only need to show that (1) \Rightarrow (4). Let $P'_1 = P_1 H^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P'_2 = P_2 H^{-1} = \begin{pmatrix} A & b \end{pmatrix}$ as in the proof of this direction in Theorem 3.1. If P_1 and P_2 are non-coincident in (1), then P'_1 and P'_2 are also non-coincident. If A is nonsingular then $b \neq 0$. If A is singular, then $b \neq 0$ because $\text{rank}(P'_2) = 3$. Now using the last part of Lemma 3.3, we can turn the reconstruction $(P'_1, P'_2, \{Hw_i\})$ into a finite reconstruction with non-coincident cameras with the first camera equal to $\begin{pmatrix} I & 0 \end{pmatrix}$. This is the statement in (4). \square

Theorem 3.5. *The four statements in Theorem 3.1 are equivalent if we replace “cameras P_1, P_2 ” in each statement with “coincident cameras P_1, P_2 ”.*

Proof: Again, we only need to prove that (1) \Rightarrow (4). If P_1, P_2 are coincident cameras in (1), then $P'_1 = P_1 H^{-1} = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P'_2 = P_2 H^{-1} = \begin{pmatrix} A & b \end{pmatrix}$ are also coincident. Therefore, $\hat{0}$ is their common center and hence $b = 0$. This implies that A is nonsingular since otherwise $\text{rank}(P'_2) < 3$. Now consider the points w'_i obtained by setting the last coordinate of each w_i from the reconstruction in (1) to 1. Then $(P'_1, P'_2, \{w'_i\})$ is a finite reconstruction of $\{(x_i, y_i)\}$. \square

By the above results we can always obtain a finite projective reconstruction whenever a projective reconstruction exists. Also, if the projective reconstruction was with non-coincident (coincident) cameras there is also a finite reconstruction with non-coincident (coincident) cameras. Further, in each case the first camera can be assumed to be $\begin{pmatrix} I & 0 \end{pmatrix}$. This understanding will be useful in the next section.

We end this section by discussing the geometry of the point pairs for which a projective reconstruction with coincident cameras exists.

Definition 3.6. Given $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \dots, m$, we say that $\{x_i\}$ is *projectively equivalent* to $\{y_i\}$ if there is a nonsingular matrix $H \in \mathbb{R}^{3 \times 3}$ such that $H\hat{x}_i \sim \hat{y}_i$ for all $1 \leq i \leq m$.

The following result captures the close relationship between projectively equivalent point sets and projective reconstruction with coincident cameras.

Theorem 3.7. *Let $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \dots, m$ be given. Then there exists a finite reconstruction $(P_1 = \begin{pmatrix} I & 0 \end{pmatrix}, P_2, \{\hat{w}_i\}_{i=1}^m)$ of $\{(x_i, y_i)\}$ where P_1 and P_2 are coincident cameras if and only if $\{x_i\}$ is projectively equivalent to $\{y_i\}$.*

Proof: Suppose $P_2 = \begin{pmatrix} A & b \end{pmatrix}$. If P_1 and P_2 are coincident, then their common camera center is $\hat{0}$ which is finite. Hence P_2 is a finite camera and $b = 0$. Unwinding $P_1 \hat{w}_i \sim \hat{x}_i$ and $P_2 \hat{w}_i \sim \hat{y}_i$ we obtain $A \hat{x}_i \sim \hat{y}_i$ for all $i = 1, \dots, m$.

For the converse, suppose there exists a nonsingular matrix $H \in \mathbb{R}^{3 \times 3}$ such that $H\hat{x}_i \sim \hat{y}_i$ for all $i = 1, \dots, m$. Then setting $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 := \begin{pmatrix} H & 0 \end{pmatrix}$, and using the notation $\hat{\hat{a}}$ for $(\hat{a}^\top, 1)^\top$ where $a \in \mathbb{R}^2$, we see that $(P_1, P_2, \{\hat{\hat{x}}_i\}_{i=1}^m)$ is a projective reconstruction of $\{(x_i, y_i)\}$ with two coincident cameras. \square

4. MAIN THEOREM

We now come to the more general situation of reconstruction. In this case, there is a distinguished fundamental matrix associated to the point pairs coming from the cameras in the reconstruction. We remark that the some results in this section are formally or informally stated in [2], but we prove them using linear algebra instead of classical projective geometry.

Theorem 4.1. *Let $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \dots, m$ be given. Consider two finite cameras $P_1 := \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 := \begin{pmatrix} A & b \end{pmatrix}$. Suppose that there exist $w_i \in \mathbb{R}^3$ ($1 \leq i \leq m$) such that $(P_1, P_2, \{\hat{w}_i\})$ is a reconstruction of $\{(x_i, y_i)\}$. Then there is a fundamental matrix associated to $\{(x_i, y_i)\}$.*

Proof: Suppose that P_1 and P_2 are non-coincident cameras. Since A is nonsingular one has $b \neq 0$. Define $F := [b]_{\times} A$. Since $b \neq 0$, $\text{rank}([b]_{\times}) = 2$ and $\text{rank}(F) = 2$. For a fixed i , the relations $P_1 \hat{w}_i \sim \hat{x}_i$ and $P_2 \hat{w}_i \sim \hat{y}_i$ imply that $\lambda_i A \hat{x}_i + b = \mu_i \hat{y}_i$ for some $\lambda_i \neq 0$, $\mu_i \neq 0$. Hence, F satisfies the epipolar constraints involving x_i and y_i :

$$\hat{y}_i^{\top} F \hat{x}_i \sim (\lambda_i (A \hat{x}_i)^{\top} + b^{\top}) [b]_{\times} A \hat{x}_i \sim (A \hat{x}_i)^{\top} [b]_{\times} A \hat{x}_i = (A \hat{x}_i)^{\top} (b \times A \hat{x}_i) = 0.$$

If P_1 and P_2 are coincident, then there is a nonsingular matrix H such that $Hx_i \sim y_i$ for all $1 \leq i \leq m$, by Theorem 3.7. Let t be any nonzero vector in \mathbb{R}^3 . It follows that for any $i = 1, \dots, m$,

$$y_i^{\top} [t]_{\times} H x_i = x_i^{\top} [t]_{\times} y_i = 0.$$

Thus $[t]_{\times} H$ is a fundamental matrix associated to $\{(x_i, y_i)\}$. \square

We now introduce a regularity condition on $\{(x_i, y_i)\}_{i=1}^m$ that is necessary for the existence of a projective reconstruction with non-coincident cameras. We will see that when the point pairs (x_i, y_i) are regular, a reconstruction with non-coincident cameras exists if and only if a fundamental matrix exists.

Definition 4.2. *Let $A \in \mathbb{R}^{3 \times 3}$ and $b \in \mathbb{R}^3$. We say that $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ is (A, b) -irregular if one of the following mutually exclusive conditions hold:*

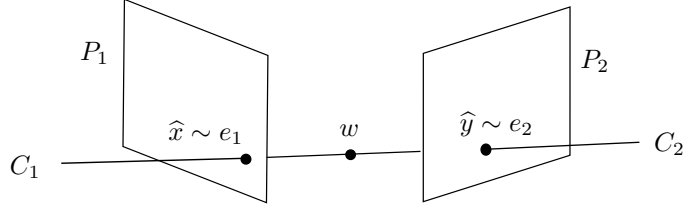
$$(4.1) \quad ([b]_{\times} A \hat{x} = 0 \text{ and } \hat{y}^{\top} [b]_{\times} \neq 0) \quad \text{or} \quad ([b]_{\times} A \hat{x} \neq 0 \text{ and } \hat{y}^{\top} [b]_{\times} = 0).$$

Say (x, y) is (A, b) -regular if it is not (A, b) -irregular.

If (x, y) is an (A, b) -irregular pair then $\hat{y}^{\top} [b]_{\times} A \hat{x} = 0$. This implies that if $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ are non-coincident finite cameras then (x, y) satisfies the epipolar constraint $\hat{y}^{\top} F \hat{x} = 0$ (where $F = [b]_{\times} A$) whether or not there is a reconstruction $w \in \mathbb{P}^3$ of (x, y) . In fact, more is true.

Since $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ is non-coincident with P_1 , one has $b \neq 0$. Since the fundamental matrix $F := [b]_{\times} A$ has rank two, both the left and right kernel of F are one-dimensional. Let $e_1, e_2 \in \mathbb{R}^3 \setminus \{0\}$ be a basis vector of the right and left kernel of F respectively. Then e_1 is called an *epipole* of P_1 while e_2 is called an epipole of P_2 . It is known that $P_1 c_2 \sim e_1$ and $P_2 c_1 \sim e_2$, where $c_1 = \hat{0}$ and $c_2 = (-A^{-1}b^{\top}, 1)^{\top}$ are the camera centres of P_1 and P_2 respectively. This implies we can take $e_1 := A^{-1}b$ and $e_2 := b$.

Suppose (x, y) is (A, b) -irregular. Then as we saw earlier, $\hat{y}^{\top} F \hat{x} = 0$ holds. If $[b]_{\times} A \hat{x} = 0$ and $\hat{y}^{\top} [b]_{\times} \neq 0$ then \hat{x} is an epipole of P_1 but \hat{y} is not an epipole of P_2 . If $[b]_{\times} A \hat{x} \neq 0$ and $\hat{y}^{\top} [b]_{\times} = 0$ holds then \hat{x} is not an epipole of P_1 but \hat{y} is an epipole of P_2 . On the other hand, we see from Figure 1 that if (P_1, P_2, \hat{w}) is a

FIGURE 1. The tuple (P_1, P_2, \hat{w}) reconstructs (x, y) .

reconstruction of (x, y) for some $w \in \mathbb{R}^3$, and if \hat{x} is the epipole of P_1 , then \hat{y} has to be the epipole of P_2 (the epipoles of the two cameras lie on the line connecting the centers of the two cameras.) This means if (x, y) is (A, b) -irregular, then there is no finite reconstruction for (x, y) using P_1, P_2 , even though the epipolar constraint is trivially satisfied. This proves the following lemma.

Lemma 4.3. *Suppose that $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ are two non-coincident finite cameras. Then, if (x, y) is (A, b) -irregular, then there is no $w \in \mathbb{R}^3$ such that (P_1, P_2, \hat{w}) is a reconstruction of (x, y) .*

Notice that Lemma 4.3 can also be verified using a simple algebraic computation, without using the notion of an epipole and the help of Figure 1.

The following two lemmas will be used to prove the main theorem of this section.

Lemma 4.4. *Suppose that $P_1 = \begin{pmatrix} I & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} A & b \end{pmatrix}$ are two non-coincident finite cameras. Then, if (x, y) is (A, b) -regular and $\hat{y}^\top [b]_\times A \hat{x} = 0$, then there exists $w \in \mathbb{P}^3$ such that (P_1, P_2, w) is a reconstruction of (x, y) .*

Proof: The assumptions about P_1 and P_2 , and the equation $\hat{y}^\top [b]_\times A \hat{x} = 0$ imply $\hat{y}, b, A \hat{x}$ are nonzero linearly dependent vectors in \mathbb{R}^3 . Thus there are scalars $\gamma, \beta, \alpha \in \mathbb{R}$, not all zero, such that

$$(4.2) \quad \gamma A \hat{x} = \beta \hat{y} - \alpha b.$$

For a scalar δ , define $w_\delta := \begin{pmatrix} \hat{x} \\ \delta \end{pmatrix}$. Then we obtain

$$P_1 w_\delta = \hat{x}, \quad \text{and} \quad P_2 w_\delta = A \hat{x} + \delta b.$$

There are three cases to consider.

Case 1: $\gamma = 0$.

Then $\hat{y} \sim b$. If $A \hat{x} = 0$, then $P_2 w_\alpha = \beta \hat{y} \sim \hat{y}$ so (P_1, P_2, w_α) is a reconstruction of (x, y) . If $A \hat{x} \neq 0$, then $\hat{y} \sim A \hat{x}$ by the regularity of (x, y) . Thus $P_2 w_0 = A \hat{x} \sim \hat{y}$ so (P_1, P_2, w_0) is a reconstruction of (x, y) .

Case 2: $\gamma \neq 0$ and $\beta = 0$.

In this case (4.2) gives $A \hat{x} = -\alpha b$ after scaling. If $\alpha = 0$ then $A \hat{x} = 0$ and $\hat{y} \sim b$ by the regularity of (x, y) . Thus $P_2 w_1 = b \sim \hat{y}$ which means (P_1, P_2, w_1) is a reconstruction of (x, y) . If $\alpha \neq 0$ then $A \hat{x} \neq 0$ and $A \hat{x} \sim b$. By the regularity of (x, y) , one has $\hat{y} \sim A \hat{x}$. Thus (P_1, P_2, w_0) is a reconstruction of (x, y) .

Case 3: $\gamma \neq 0$ and $\beta \neq 0$.

(4.2) implies $A \hat{x} = \beta \hat{y} - \alpha b$ after scaling. Hence $P_2 w_\alpha = A \hat{x} + \alpha b = \beta \hat{y} \sim \hat{y}$ which concludes that (P_1, P_2, w_α) is a reconstruction of (x, y) . \square

Lemma 4.5. *Let F be a fundamental matrix and let $e_2 \in \ker(F^\top) \setminus \{0\}$. Define $P := ([e_2]_\times F \quad e_2)$. Then P has rank three.*

Proof: The proof can be found in [2, page 256], but we rewrite it here for the self-containedness of this note. Since $e_2 \in \ker(F^\top) \setminus \{0\}$, we have $\text{rank}([e_2]_\times F) = 2$. It implies that the column space of $[e_2]_\times F$ is a plane in \mathbb{R}^3 . Since e_2 is a nonzero vector orthogonal to any vector in this plane, we know $\text{rank}(P) = 3$. \square

We are now ready to prove the main theorem.

Theorem 4.6. *Let $(x_i, y_i) \in \mathbb{R}^2 \times \mathbb{R}^2$, $i = 1, \dots, m$ be given. Then the following statements are equivalent:*

- (1) *There exists a finite reconstruction of $\{(x_i, y_i)\}$ where one of the cameras is $(I \quad 0)$ and the two cameras are non-coincident.*
- (2) *There is a fundamental matrix F associated to $\{(x_i, y_i)\}$ such that (x_i, y_i) is $([e_2]_\times F, e_2)$ -regular for all i , where $e_2 \in \ker(F^\top) \setminus \{0\}$.*

Proof: First we show (2) \Rightarrow (1). Let the matrix F stated in (2) be given. Notice that $([e_2]_\times F \quad e_2)$ is a camera matrix by Lemma 4.5. Then, take $a \in \mathbb{R}^3$ so that $A := [e_2]_\times F - e_2 a^\top$ is nonsingular and $P_1 := (I \quad 0)$ and $P_2 := (A \quad e_2)$ are non-coincident finite cameras; see the proof of Lemma 3.3 for how a is chosen. As $[e_2]_\times A = -e_2^\top e_2 F$, one has $y_i^\top [e_2]_\times A x_i = 0$ for all i . Then (1) holds by Theorem 3.1 and Lemma 4.4.

Next we show the converse. Assume (1) holds. Then there is a finite camera P_2 so that $P_1 := (I \quad 0)$, P_2 are non-coincident cameras, and there are $w_i \in \mathbb{R}^3$ ($1 \leq i \leq m$) such that $(P_1, P_2, \{\hat{w}_i\})$ is a reconstruction of $\{(x_i, y_i)\}$. We let $P_2 := (A \quad b)$ where $A \in \mathbb{R}^{3 \times 3}$ is nonsingular and $b \in \mathbb{R}^3 \setminus \{0\}$. Consider the fundamental matrix $F := [b]_\times A$. By Theorem 4.1 and Lemma 4.3, the epipolar constraints are satisfied and each (x_i, y_i) is (A, b) -regular. Since $F^\top = -A^\top [b]_\times$, we have $b \in \ker(F^\top) \setminus \{0\}$. Moreover, as $[b]_\times F = -b^\top b A$, we know each (x_i, y_i) is $([b]_\times F, b)$ -regular. Thus the statement (2) follows. \square

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